Math 142 Lecture 5 Notes

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1 Quotient Spaces and Introduction to Compactness

1.1 Quotient spaces (continued)

If we have a $f : X \to Y$ is surjective, then we can define a partition $\mathcal{P}$ on X by taking sets $f^{-1}(y)$ for each $y \in Y$. So we can define an identification space Y^* from X and $\mathcal{P}$. Last time, we proved the following theorem.

Theorem 1.1. *If $f : X \to Y$ is continuous, surjective, and maps open sets to open sets, then $Y^* \cong Y$.*

We said last time that a quotient space X/A was the special case where $A \subseteq X$, $Y = \{*\}$, $f : A \to Y$, and X/A is the identification space $X \cup_f Y$.

Proposition 1.1. *For any $n \in \mathbb{N}$ with $n > 1$, $B^n/S^{n-1} \cong S^n$.*

Proof. Recall from stereographic projection that, calling p the "north pole" on S^n, $S^n \setminus \{p\} \cong \mathbb{R}^n$. Let $g : \mathbb{R}^n \to S^n \setminus \{p\}$ be a homeomorphism. We also have that $B^n \setminus S^{n-1} \cong \mathbb{R}^n$ by the homeomorphism $h : B^n \setminus S^{n-1} \to \mathbb{R}$ given by $x \mapsto (1 + \tan(\|x\| \pi/2))x$. Show that this is a homeomorphism (exercise).

Then define $f : B^n \to S^n$ by

$$f(x) = \begin{cases} p & x \in S^{n-1} = \partial B^n (\text{boundary of } B^n) \\ g(h(x)) & x \notin S^{n-1} \quad (\text{i.e. } x \in (\mathring{B}^n)). \end{cases}$$

Show that f is continuous (exercise). Show that f takes open sets to open sets (also exercise, but similar to the previous). Then the previous theorem implies that Y^* from f is homeomorphic to S^n. But

$$f^{-1}(y) = \begin{cases} \text{singleton} & y \neq p \\ S^{n-1} & y = p, \end{cases},$$

so $Y^* = B^n/S^{n-1}$. ∎

1.2 Compactness

1.2.1 Open covers and compactness

Definition 1.1. An *open cover* of a topological space X is a collection[1] of open sets $\{A_i\}$ with $A_i \subseteq X$ such that $X = \bigcup_i A_i$. If $\{A_i\}$ and $\{B_j\}$ are open covers of X, and $\{B_j\} \subseteq \{A_i\}$, then $\{B_j\}$ is called a *subcover* of $\{A_i\}$.

Example 1.1. Let X be any space. Then $\{X\}$ is an open cover.

Example 1.2. Let $X = \mathbb{R}$, and take the collection $\{A_i\} := \{B_\varepsilon(x) : \varepsilon > 0, x \in \mathbb{R}\}$; this is an open cover. Let $\{B_j\} := \{B_\varepsilon(x) : \varepsilon > 0, \varepsilon \in \mathbb{Q}, x \in \mathbb{Q}\}$; then $\{B_j\}$ is a subcover of $\{A_i\}$.

Definition 1.2. A space X is *compact* if every open cover of X has a finite subcover.[2]

Example 1.3. We show that $\mathbb{R}$ with the usual topology is not compact. Let $X = \mathbb{R}$ and $A_i = (i-1, i+1)$ for each $\mathbb{Z}$. $\{A_i\}$ is an open cover of $\mathbb{R}$, but for $i \in \mathbb{Z}$, $i \in A_j \implies i = j$. So there are no subcovers of $\{A_i\}$; in particular, there are no finite subcovers. Similarly, $\mathbb{R}^n$ is not compact.

Definition 1.3. A subset $A \subseteq X$ is *compact* if it is compact with the subspace topology.

Theorem 1.2. *If $f : X \to Y$ is continuous, and X is compact, then the image $f(X)$ is compact.*

Proof. Assume that f is surjective; if not, just consider $g : X \to f(X)$ given by $g(x) = f(x)$. Let $\{A_i\}$ be an open cover of $Y = f(X)$. Since f is continuous, $f^{-1}(A_i)$ is open for each A_i, and $\forall x \in X$, $x \in f^{-1}(A_i)$ for some i (as $\{A_i\}$ is a cover for Y). So $\{f^{-1}(A_i)\}$ is an open cover of X, and by the compactness of X, there exists a finite subcover of X; i.e. $X = f^{-1}(A_{i_1}) \cup \cdots \cup f^{-1}(A_{i_n})$. Since $f(f^{-1}(A_i)) = A_i$, we have $Y = f(X) = A_{i_1} \cup \cdots \cup A_{i_n}$. So $\{A_{i_1}, \ldots, A_{i_n}\}$ is a finite subcover of $\{A_i\}$. Since $\{A_i\}$ was an arbitrary open cover, this works for every cover. Hence, Y is compact. $\square$

Here is the flow of the previous proof in a picture:

$$\begin{array}{ccc}
X & & Y \\
 & & \\
\text{open cover} & \longleftarrow & \text{open cover} \\
\downarrow & & \\
\text{finite subcover} & \longrightarrow & \text{finite subcover.}
\end{array}$$

The following theorem has a similar structure to its proof.

[1] This collection need not even be countable. We may have an uncountable collection of open sets in our cover.

[2] Compactness is a property of the space itself, not of a particular cover.

Theorem 1.3. *If X is compact, and $B \subseteq X$ is closed, then B is compact.*

Proof. Let $\{A_i\}$ be an open cover of B. Then each $A_i = A'_i \cap B$ for some $A'_i \subseteq X$ open, and $B \subseteq \bigcup A'_i$. Note that $\{A'_i\} \cup \{X \setminus B\}$ is an open cover of X; $X \setminus B$ is open because B is closed. X is compact, so there exists a finite subcover $X = A'_{i_1} \cup \cdots A'_{i_n} \cup (X \setminus B)$; the set $X \setminus B$ may not be necessary, but it has empty intersection with B, so it doesn't matter if we keep it. Then $B = A_{i_1} \cup \cdots \cup A_{i_n}$, and since $\{A_i\}$ was a generic open cover, we conclude that B is compact. □

1.2.2 Hausdorff Spaces

Definition 1.4. A space X is *Hausdorff* if for all $x, y \in X$ with $x \neq y$, there are neighborhoods U_x of x and U_y of y such that $U_x \cap U_y = \varnothing$.

Theorem 1.4. *If X is Hausdorff, and $A \subseteq X$ is compact, then A is closed.*

We will delay proof of this until next time. For now, we will use this theorem to prove the following theorem.

Theorem 1.5. *If $f : X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Proof. If f takes closed sets to closed sets, then f^{-1} is continuous, and we are done. If $B \subseteq X$ is closed, then B is compact. The function f is continuous, so $f(B) \subseteq Y$ is compact. Then, by the previous theorem, $f(B)$ is closed. □